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Existence of Maximal Solutions

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Let Ω be a plane bounded region. Let $\mathscr{U} = \{u_\mu(P):\mu(P) \in L_\infty(\Omega),\ u_\mu \in H^2_{2,0}(\Omega)$ and $a(P, \mu(P))u_{\mu,xx} + 2b(P, \mu(P))u_{\mu,xy} + c(P, \mu(P))u_{\mu,yy} = f(P)$ for $P \in \Omega\}$; here we are given $a(P, X), b(P, X), c(P, X) \in L_\infty(\Omega \times E^1)$, $f(P) \in L_p(\Omega)$ with $p > 2$, and our partial differential equation is uniformly elliptic. The functions $\mu(P)$ are called profiles. We establish sufficient conditions—which when they apply are constructive—that there exist a $\mu_0 \in L_\infty(\Omega)$ such that $u_{\mu_0}(P) \geqslant u_\mu(P)$ for all $P \in \Omega$ and for each $\mu \in L_\infty(\Omega)$. Similar results are obtained for a difference equation and convergence is proved.

1. INTRODUCTION

Let Ω be a region of the plane and let $L_\infty(\Omega)$ be the set of equivalence classes of bounded measurable functions on Ω. Let $P = (x, y)$ be a generic point of Ω.

Let $a(P, X)$, $b(P, X)$, and $c(P, X)$ be given functions which satisfy the following condition:

(I) The functions $a(P, X)$, $b(P, X)$, and $c(P, X) \in C(\bar{\Omega} \times E^1)$, where E^n is Euclidean n-space. Moreover, for every real $(\xi_1, \xi_2) \in E^2$ and for every $(P, X) \in \bar{\Omega} \times E^1$, there exist real positive and finite constants λ_0 and λ_1—which are independent of P, X, ξ_1, ξ_2—such that

$$\text{(a)}\quad \lambda_0(\xi_1^2 + \xi_2^2) \leqslant a(P, X)\,\xi_1^2 + 2b(P, X)\,\xi_1\xi_2 + c(P, X)\,\xi_2^2 \leqslant \lambda_1(\xi_1^2 + \xi_2^2)$$

and

$$\text{(b)}\quad \lambda_0 > 2b(P, X) \geqslant 0$$

for all $P \in \Omega$ and for all $X \in E^1$.

Let $\mu(P) \in L_\infty(\Omega)$. If (I) is satisfied, then there exists one and only one $u_\mu(P) \in H^2_{2,0}(\Omega)$ such that

$$\begin{gathered} a(P, \mu(P))\, u_{\mu,xx} + 2b(P, \mu(P))\, u_{\mu,xy} + c(P, \mu(P))_{\mu,yy} = f(P), \quad P \in \Omega, \\ u_\mu(P) = 0, \qquad P \in \partial\Omega, \end{gathered} \tag{1.1}$$

when we assume that $f(P) \in L_p(\Omega)$ with $p \geqslant 2$; Talenti [9]. The partial derivatives of $u_\mu(P)$ are with respect to the components of P and the inclusion of μ in the notation $u_\mu(P)$ is to emphasize the dependence of a solution to (1.1) on the values of μ. The function μ is called a *profile* and u_μ is called the solution associated with the profile μ.

The existence of a solution to (1.1)—as proved by Talenti—does not require the condition I(b). However we will consider a difference equation associated with (1.1) and for the methods of this paper to apply to that discrete problem we must require I(b) so as to be assured that our difference equation is antitone.

Suppose there exists an element $\mu_0 \in L_\infty(\Omega)$ such that, for each $\mu \in L_\infty(\Omega)$,

$$u_{\mu_0}(P) \geqslant u_\mu(P) \tag{1.2}$$

for every $P \in \Omega$. Then we call μ_0 a *maximal profile* and the solution associated with μ_0 is called a *maximal solution*.

Our purpose in this paper is to establish sufficient conditions that (1.1) has a maximal solution and that a discrete equation—as yet unspecified—associated with (1.1) has a maximal solution. We will also derive conditions for the convergence of the discrete maximal solutions to a solution of (1.2); hence maximal solutions to the discrete problem will approximate a maximal solution of (1.1).

The next requirement we impose on the principal coefficients of (1.1) will allow us to restrict the profiles so that their range is in a fixed compact subset:

(II) There exists two finite and distinct real numbers A and B such that for each $P \in \Omega$ and for each $X \in E^1$ there is an $Y \in [A, B] \subset E^1$—this interval may be an open, half-open or closed interval—for which $a(P, X) = a(P, Y)$, $b(P, X) = b(P, Y)$, and $c(P, X) = c(P, Y)$.

This condition is satisfied if, for example, each of the three functions are periodic functions of X but not necessarily of the same period.

Let $\mathscr{H}^0 = \{\mu(P)\colon \mu(P) \in L_\infty(\Omega)$ and $A \leqslant \mu(P) \leqslant B$ a.e.$\}$, let $\mathscr{U} = \{u_\mu\colon u_\mu \in H^2_{2,0}(\Omega)$ solves (1.1) and $\mu \in L_\infty(\Omega)\}$ and let $\mathscr{U}_0 = \{u_\mu\colon \mu(P) \in \mathscr{H}^0$ and $u_\mu \in H^2_{2,0}(\Omega)$ solves (1.1$\}$. If the coefficients in (1.1) satisfy the conditions (I) and (II)—but we do not need (b) here, then $\mathscr{U} = \mathscr{U}_0$. This is obviously true if $\mu(P)$ is a step-function on Ω. Now the result follows since convergence a.e.

of $\mu_n(P) \in L_\infty(\Omega)$ to $\bar{\mu}(P)$ implies that u_{μ_n} converge strongly in $H^2_{2,0}(\Omega)$ to $u_{\bar{\mu}}$. Therefore our search for a maximal profile may be restricted to $\mathscr{H}^0$.

In order to obtain an existence theorem for a maximal solution—both for (1.1) and our discrete equation associated with (1.1)—we must make a further assumption on the behavior of the principal coefficients.

(III) Let $P \in \Omega$. Then, for each of the eight possible combinations of the terms $\max\{a(P, Y): Y \in [A, B]\}$ or $\min\{a(P, Y): Y \in [A, B]\}$, and $\max\{b(P, Y): Y \in [A, B]\}$ or $\min\{b(P, Y): Y \in [A, B]\}$, and $\max\{c(P, Y): Y \in [A, B]\}$ or $\min \{c(P, Y): Y \in [A, B]\}$, there exist an $X \in [A, B]$ such that $a(P, X)$, $b(P, X)$, and $c(P, X)$ assumes the corresponding value of the particular combination; the interval $[A, B]$ may be open, half-open or closed. In this condition we are requiring that, for any $P \in \Omega$, there exists an $X_0 \in [A, B]$ such that $a(P, X_0) = \max\{a(P, Y): Y \in [A, B]\}$, $b(P, X_0) = \max\{b(P, Y): Y \in [A, B]\}$, $c(P, X_0) = \max\{c(P, Y): Y \in [A, B]\}$ and there exists an $X_1 \in [A, B]$ such that $a(P, X_1) = a(P, X_0)$, $b(P, X_1) = b(P, X_0)$, $c(P, X_1) = \min\{c(P, Y): Y \in [A, B]\}$ and there exists $X_2 \in [A, B]$ such that $a(P, X_2) = a(P, X_0)$, $b(P, X_2) = \min\{b(P, Y): Y \in [A, B]\}$, $c(P, X_2) = c(P, X_1)$, etc.

If $a(P, X) = 4 + \cos X$, $b(P, X) \equiv 0$, and $c(P, X) = 1$, then condition (III) is satisfied with $[A, B] = [0, \pi]$. This is the smallest interval over which (III) is satisfied but any finite interval containing $[0, \pi]$ would do.

We will also prove the existence of maximal solutions, both for (1.1) and a difference equation associated with it, when we replace (III) by the condition (IV); we will assume that $b(P, X) \equiv 0$ as it will simplify our computations and the extension to the general case will be easy.

(IV) Let $a(P, X)$, $c(P, X) \in C^2(E^1)$ for each $P \in \Omega$. Then there exists an interval $[A, B]$—which may be taken as open, half-open or closed interval—such that the following conditions are satisfied:

(i) For each $(P, X) \in \Omega \times [A, B]$, $|a'(P, X)| + |c'(P, X)| > 0$;

(ii) for each $P \in \Omega$ there exists X_1, X_2, X_3, and X_4 in the interior of the interval $[A, B]$ such that $a(P, X_1) = \max\{a(P, Y): Y \in [A, B]\}$, $a(P, X_2) = \min\{a(P, Y): Y \in [A, B]\}$, $c(P, X_3) = \max\{c(P, Y): Y \in [A, B]\}$, and $c(P, X_4) = \min\{c(P, Y): Y \in [A, B]\}$;

(iii) if, for $X, Y \in [A, B]$, we have that $a'(P, X)\, c'(P, X)\, a'(P, Y)\, c'(P, Y) \neq 0$ and $a'(P, X)\, c'(P, Y) = a'(P, Y)\, c'(P, X)$, then $X = Y$;

(iv) if, for $X, Y \in [A, B]$, $a'(P, X) = a'(P, Y) = 0$, and $a''(P, X)\, a''(P, Y) \geqslant 0$, then $X = Y$;

(v) if, for $X, Y \in [A, B]$, $c'(P, X) = c'(P, Y) = 0$, and $c''(P, X)\, c''(P, Y) \geqslant 0$, then $X = Y$;

(vi) for each $P \in \Omega$ there is an $\delta = \delta(P) > 0$ such that, for all $\lambda \in [-1, 1]$ and $X \in [A, B]$.

$$| a'(P, X) c''(P, X + \lambda\delta) - a''(P, X + \lambda\delta) c'(P, Y)| > 0.$$

The conditions (I), (II), and (IV) are satisfied if, for example, $a(P, X) = 2 + \sin X$, $b(P, X) \equiv 0$, and $c(P, X) = 2 + \cos X$; here $[A, B] = [0, 2\pi)$ and, for each $P \in \Omega$, $\delta(P)$ is any element of $(0, \pi/2)$. These functions do not satisfy the condition (III). Hence, in some sense, the condition in (IV) is complementary to the condition in (III).

In conditions (II), (III), and (IV) we are not necessarily requiring that the interval $[A, B]$ be the smallest interval satisfying these conditions. Hence, when we assume that (II) and (III) or (II) and (IV) are satisfied, we simply seek a finite interval $[A, B]$ over which these conditions hold.

The methods of this paper require (III) or (IV). These conditions are both general and restrictive. Since our methods—when they apply—are constructive, it is not unreasonable that the hypotheses are restrictive. However we do believe—based on some computer experimentation—that our results are valid under conditions much less confining than (III) or (IV).

Our motivation for the study of problems of this type—where the control appears in the principal coefficients of the governing equation—is provided by concrete engineering problems; e.g. certain optimization problems in hydrodynamic lubrication and structural dynamics.

The first to develop the idea of a maximal solution was C. Pucci in [7]. In that paper he proved in an eloquent manner the following result: There exists a triplet of functions $(\bar{a}(P), \bar{b}(P), \bar{c}(P)) \in \mathscr{L}_\alpha = \{(a(P), b(P), c(P)) \in L_\infty(\Omega)$ with $\alpha \mid \xi \mid^2 \leqslant a(P)\, \xi_1^2 + 2b(P)\, \xi_1\xi_2 + c(P)\, \xi_2^2 \leqslant \mid \xi \mid^2 = \xi_1^2 + \xi_2^2$ for $(\xi_1, \xi_2) \in E^2\}$ such that every solution $u = u(a, b, c: P) \in H_2^2(\Omega)$ of the equation

$$au_{,xx} + 2bu_{,xy} + cu_{,yy} = 0, \qquad P \in \Omega,$$
$$u(P) = \phi(P), \qquad P \in \partial\Omega,$$

with $(a, b, c) \in \mathscr{L}_\alpha$, is such that

$$u(\bar{a}, \bar{b}, \bar{c}: P) \geqslant u(a, b, c: P)$$

for all $P \in \Omega$ and for all $(a, b, c) \in \mathscr{L}_\alpha$. The methods of [7] are not constructive and in our problem we are given a much more specific class of control functions.

In Section 2 we assume that conditions (I), (II), and (III) are satisfied and that Ω is a rectangular region. We associate with (1.2) a discrete problem and derive sufficient conditions that a discrete maximal solution exist. When these conditions are satisfied we obtain the discrete maximal solution by an iterative method. We also discuss a modification of the iterative method so that the discrete profiles in the iteration converge. The condition in I(b) is only to assure that the difference problem is antitone. We could allow for the case that $\lambda_0 > 2 \mid b(P, X)|$ but this would require a change in the formulation of the mixed second order difference quotient depending on the sign of $b(P, X)$. The restriction that Ω be a rectangular region is due to the fact that our methods require an estimate on the discrete $L_2(\Omega)$ norm of second order difference quotients. These estimates were first derived in [6] for rectangular regions and later in [3] for a circular region but using unbalanced difference quotients. Additional comments on the dependence on the geometry of Ω are given at the end of this section.

In Section 3 we show how our method for the discrete problem may be extended to the problem in (1.2) when the governing equation is given by (1.1). We then show that maximal solutions of our discrete problem converge to a maximal solution of (1.2). The existence of a maximal solution for the differential equation is over more general regions than rectangles because the $L_2(\Omega)$ estimates on second order derivatives is well established.

In Section 4 we consider the discrete problem in Section 2 when the conditions (I), (II), and (IV) are satisfied. The treatment of the differential equation satisfying these conditions is not presented as it follows closely the treatment in Section 3 with modifications given in Section 4.

In Section 5 two numerical examples are given.

It is possible that a given region Ω may be decomposed into two regions such that (III) is satisfied over one region and (IV) is satisfied over the other region. Our methods may be modified so as to cover this case as well as the presence of lower order terms in (1.1). It is also clear that what we say about maximal solutions may also be said—with some slight changes—for "minimal solutions."

2. A Discrete Problem when (III) is Satisfied

In this section we will establish criteria that a discrete problem associated with (1.2) has a solution whenever the principal coefficients (i.e., the coefficients of the second order derivatives) satisfy the conditions in (I), (II), and (III).

We assume that Ω is the rectangular region determined by the vertices $(0, 0)$, $(a, 0)$, (a, b), and $(0, b)$. The extension to more general regions will be discussed at the end of this section.

Place a square grid on the plane. If $P = (x, y)$ is a grid point, then the neighbors of P are the points $P_1 = (x + h, y)$, $P_2 = (x, y + h)$, $P_3 = (x - h, y)$, $P_4 = (x, y - h)$, and $P_5 = (x + h, y + h)$. Let Ω_h be the set of mesh points $P \in \Omega$ such that all the neighbors of P are in $\bar{\Omega}$ and let $\partial\Omega_h$ be the set of mesh points in $\bar{\Omega}$ with at least one neighbor in the exterior of $\bar{\Omega}$. We assume—with no loss in generality—that $h > 0$ is so selected that the $\partial\Omega_h$ is contained in the $\partial\Omega$.

Let s_i, $i = 1, 2, 3, 4$, be the sides of the $\partial\Omega$ with s_1 on the line $x = a$, s_2 on the line $y = b$, etc. Those mesh points on s_i will be denoted by $s_{(h)i}$.

Let $\mathscr{H}_h{}^0 = \{W(P)\colon W(P)$ is defined on Ω_h and $A \leqslant W(P) \leqslant B$ for each $P \in \Omega_h\}$; this is our space of *admissible discrete profiles*. The difference problem we use to approximate (1.1) is

$$\begin{aligned} a(P, W(P))\, U_{x\bar{x}}(P) + 2b(P, W(P))\, U_{xy}(P) + c(P, W(P))\, U_{y\bar{y}}(P) \\ = f(P), \qquad P \in \Omega_h, \\ U(P) = 0, \qquad P \in \partial\Omega_h. \end{aligned} \tag{2.1}$$

This equation is obtained from (1.1) by replacing $u_{,xx}(P)$ by $U_{x\bar{x}}(P) = \{U(P_1) + U(P_3) - 2U(P)\}/h^2$, $u_{,yy}(P)$ by $U_{y\bar{y}}(P) = \{U(P_2) + U(P_4) - 2U(P)\}/h^2$, and $u_{,xy}(P)$ by $U_{xy}(P) = \{U(P_5) - U(P_2) - U(P_1) + U(P)\}/h^2$. Note that U_x denotes a forward difference quotient and $U_{\bar{x}}$ denotes a backward difference quotient.

In order to simplify our notation we define the operator

$$\begin{aligned} \mathscr{L}_h(W)[V(P)] = a(P, W(P))\, V_{x\bar{x}}(P) + 2b(P, W(P))\, V_{xy}(P) \\ + c(P, W(P))\, V_{y\bar{y}}(P) \end{aligned} \tag{2.2}$$

where W is an arbitrary element of $\mathscr{H}_h{}^0$ and V is an arbitrary element of $\mathscr{A}_h = \{V\colon V(P)$ is everywhere defined and finite on Ω_h, and $V(P) = 0$ for $P \in \partial\Omega_h\}$.

LEMMA 1. *Let the principal coefficients satisfy the condition* (I) *and assume that $f(P)$ is finite for each $P \in \Omega_h$. Let $W(P)$ be an element in $\mathscr{H}_h{}^0$. Then there exists one and only one element $U_W \in \mathscr{A}_h$* such that

$$\mathscr{L}_h(W)[U_W(P)] = f(P) \tag{2.3}$$

for all $P \in \Omega_h$. Moreover the operator $\mathscr{L}_h(W)[V]$ is antitone for each $W \in \mathscr{H}_h{}^0$ and for $V \in \mathscr{A}_h$; i.e., $\mathscr{L}_h(W)[V] \leqslant 0$ at each $P \in \Omega_h$ implies that $V(P) \geqslant 0$ at each $P \in \Omega_h$.

Proof. The condition in (I) assures us that the matrix associated with $-\mathscr{L}_h(W)[V]$ is monotone. Our result now follows.

We remark that the existence and uniqueness part of *Lemma* 1 does not require I(b) but that condition is necessary for the antitone property.

Observe that the notation for a solution to (2.3), U_W, carries with it the notation for the discrete profile which is used in the principal coefficients of (2.1). In general, if two discrete profiles, W_1 and W_2, differ at only one point, then the corresponding solutions to (2.3), U_{W_1} and U_{W_2}, will differ at all points of Ω_h.

If $V(P) \in \mathscr{A}_h$, then $\| V \|^2$ is defined as

$$\| V \|^2 = h^2 \sum_{P \in \Omega_h} V^2(P)$$

and $\| V \|_2^2$ is defined as

$$\| V \|_2^2 = \| V_{x\bar{x}} \|^2 + 2 \| V_{xy} \|^2 + \| V_{y\bar{y}} \|^2;$$

here $\| V_{xy} \|_2^2 = h^2 \sum_{\Omega_h'} | V_{xy} |^2$ with $\Omega_h' = \Omega_h + s_{(h)3} + s_{(h)4}$.

LEMMA 2. *If the hypotheses of* Lemma 1 *are satisfied—but we do not need* (b) *of condition* (I), *then, for each and every* $W \in \mathscr{H}_h^0$,

$$\| U_W \|_2^2 \leqslant 2\lambda_1^2 \| f \|^2 / \lambda_0^4. \tag{2.5}$$

Moreover, for each and every $W \in \mathscr{H}_h^0$,

$$\max_{P \in \Omega_h} | U_W(P) | \leqslant a^{1/2} b^{1/2} \| U_W \|_2 . \tag{2.6}$$

Proof. The result in (2.5) is proved in [4, p. 365] for our difference approximation. Although (2.6) is also proved there we will give here an easier derivation. Let $P = (x, y) \in \Omega_h$. Then

$$| U_W(P) | = \left| h \sum_{\gamma=0}^{x-h} U_{Wx}(\gamma, y) \right| \leqslant a^{1/2} \left(h \sum_{\gamma=0}^{a-h} | U_{Wx}(\gamma, y) |^2 \right)^{1/2}$$

and

$$| U_{Wx}(\gamma, y) | = \left| h \sum_{\kappa=0}^{y-h} U_{Wxy}(\gamma, \kappa) \right| \leqslant b^{1/2} \left(h \sum_{\kappa=0}^{b-h} | U_{Wxy}(\gamma, \kappa) |^2 \right)^{1/2}.$$

Hence,

$$h \sum_{\gamma=0}^{a-h} | U_{Wx}(\gamma, y) |^2 \leqslant bh^2 \sum_{\Omega_h'} | U_{Wxy} |^2$$

and the result follows.

The discrete problem associated with (1.3) is the following: Find an element $W_0 \in \mathscr{H}_h^0$ such that

$$U_{W_0}(P) \geqslant U_W(P)$$

for each $P \in \Omega_h$ and for every $W \in \mathscr{H}_h^0$. We call W_0 a *discrete maximal profile* and we call U_{W_0} a *discrete maximal solution.* By our uniqueness result a discrete maximal profile determines a discrete maximal solution.

LEMMA 3. *If the principal coefficients of* (2.1) *satisfy the conditions in* (I) *and* (II), *then to every* $X(P) \in \{X: X(P)$ *is defined and finite at each* $P \in \Omega_h\}$ *there exists at least one* $W(P) \in \mathscr{H}_h^0$ *such that*

$$U_W(P) = U_X(P)$$

for all $P \in \Omega_h$.

Proof. By condition (II), to every value of $X(P)$ there exists a number $Y \in [A, B]$ such that $a(P, Y) = a(P, X(P))$ and $b(P, Y) = b(P, X(P))$ and $c(P, Y) = c(P, X(P))$. Now define $W(P) = Y$. Since this process does not change the coefficients, we get the same solution.

Let

$$\mathscr{S}_h = \{\xi(P): \xi \in \mathscr{A}_h \text{ and } \| \xi \|_2^2 \leqslant 2\lambda_1^2 \|f\|^2/\lambda_0^4\}.$$

We know that if a maximal solution exists, then it is in $\mathscr{S}_h$.

Let L_h be an operator defined on $\mathscr{A}_h$ as

$$L_h[V(P)] = V_{x\bar{x}}(P) + \alpha V_{xy}(P) + V_{y\bar{y}}(P), \tag{2.7}$$

where $\alpha \in [0, 1)$ is some constant.

The next result follows easily.

LEMMA 4. *Let* α *and* γ *be positive numbers with* $\alpha \in [0, 1)$ *and assume the conditions in* (I) *and* (II) *are satisfied. Then for each* $\xi \in \mathscr{A}_h$ *and for each* $W \in \mathscr{H}_h^0$ *there exists a unique* $Z(P) \equiv Z(\xi; W : P)$ *such that*

$$\begin{aligned} \gamma L_h[Z(P)] &= \gamma L_h[\xi(P)] - \mathscr{L}_h(W)[\xi(P)] + f(P), \qquad P \in \Omega_h, \\ Z(P) &= 0, \qquad P \in \partial\Omega_h ; \end{aligned} \tag{2.8}$$

the operator L_h *is antitone for each* $\alpha \in [0, 1)$. *For each* $Q \in \Omega_h$ *there exists a unique* $G(P; Q)$ *such that* $L_h[G(P; Q)] = -\delta(P; Q)/h^2$ *for* $P \in \Omega_h$, $G(P; Q) = 0$ *for* $P \in \partial\Omega_h$; *here* $\delta(P; Q) = 0$ *unless* $P = Q$ *whence* $\delta(P; P) = 1$. *If* $G(P; Q)$ *is the discrete Green's function for* L_h *in* (2.8), *then*

$$Z(\xi; W : P) = -(1/\gamma)\, h^2 \sum_{Q \in \Omega_h} G(P; Q)\{\gamma L_h[\xi(Q)] - \mathscr{L}_h(W)[\xi(Q)] + f(Q)\}. \tag{2.9}$$

Moreover,

$$\| Z \|_2^2 \leqslant C_0^2(C_1^2 \| \xi \|_2^2 + \|f\|^2/\gamma^2),$$

where

$$C_0^2 = 8(1 + \alpha/2)^2/(1 - \alpha/2)^4$$

and

$$C_1 = \max\{|\, 1 - a(P, X)/\gamma \,|, |\, \alpha - 2b(P, X)/\gamma \,|, |\, 1 - c(P, X)/\gamma \,| : (P, X) \in \Omega_h \times [A, B]\}. \tag{2.10}$$

Hence, if $C_0^2(2C_1^2\lambda_1^2/\lambda_0^4 + 1/\gamma^2) < 2\lambda_1^2/\lambda_0^4$ and $\xi \in \mathscr{S}_h$, then $Z \in \mathscr{S}_h$.

Let Q be an element of Ω_h and let $\xi \in \mathscr{A}_h$. Then define $\mathscr{W}(\xi : Q) = \{W: W \in [A, B]$ and $\gamma L_h[\xi(Q)] - \mathscr{L}_h(W)[\xi(Q)] = \min\{\gamma L_h[\xi(Q)] - \mathscr{L}_h(X)[\xi(Q)]: X \in [A, B]\}\}$ and let $\mathscr{W}(\xi) = \{W: W \in \mathscr{H}_h^0$ and, for each $Q \in \Omega_h$, $W(Q) \in \mathscr{W}(\xi : Q)\}$. Elements of $\mathscr{W}(\xi)$ will be denoted by $W(\xi : \cdot)$ or $W(\xi)$.

LEMMA 5. *If conditions* (I) *and* (II) *are staisfied by* (2.1), *then* $\mathscr{W}(\xi : Q) \neq 0$ *for every* $\xi \in \mathscr{A}_h$ *and for every* $Q \in \Omega_h$. *Hence there exists at least one element* $W_0 \in \mathscr{W}(\xi)$ *such that*

$$Z(\xi; W_0 : P) \geqslant Z(\xi; W : P)$$

for each $P \in \Omega_h$ *and for each* $W \in \mathscr{H}_h^0$.

Proof. One need only observe that $\mathscr{L}_h(X)[\xi]$ is a continuous function of $X \in E^1$.

Now we will study the behavior of $Z = Z(\xi; W_0(\xi) : P)$ as a function of $\xi \in \mathscr{A}_h$; note that $W_0(\xi) \in \mathscr{W}(\xi)$. We will show that Z is a Lipschitz function of $\xi \in \mathscr{A}_h$. To do this we must require that the principal coefficients satisfy additional properties; these are contained in (III).

Let $\xi \in \mathscr{A}_h$ and let $Q \in \Omega_h$. Suppose that $\xi_{x\bar{x}}(Q) > 0$, $\xi_{xy}(Q) < 0$, and $\xi_{y\bar{y}}(Q) > 0$. Then, by (III), there exists an $W \in \mathscr{W}(\xi : Q)$ such that $\gamma - a(Q, W) = \max\{\gamma - a(Q, X): X \in [A, B]\}$, $\alpha\gamma - 2b(Q, W) = \min\{\alpha\gamma - 2b(Q, X): X \in [A, B]\}$, and $\gamma - c(Q, W) = \max\{\gamma - c(Q, X): X \in [A, B]\}$. If, for some $\eta \in \mathscr{A}_h$, $\eta_{x\bar{x}}(Q) = 0$, $\eta_{xy}(Q) < 0$, and $\eta_{y\bar{y}}(Q) > 0$, then $\mathscr{W}(\xi : Q) \subset \mathscr{W}(\eta : Q)$ but these sets need not be equal.

The next result is central to the development of this section.

LEMMA 6. *Let the conditions in* (I), (II), *and* (III) *be satisfied. Then* $Z(\xi; W_0(\xi) : P)$ *is a Lipschitz function of* $\xi \in \mathscr{A}_h$. *In particular, if* $\xi_1, \xi_2 \in \mathscr{A}_h$ *and if* $Z_1 = Z(\xi_1; W_0(\xi_1) : P)$, $Z_0 = Z(\xi_2; W_0(\xi_2) : P)$, *then*

$$\| Z_1 - Z_0 \|_2^2 \leqslant C_2^2 \| \xi_1 - \xi_2 \|_2^2, \tag{2.11}$$

where

$$C_2^2 = 6C_1^2C_0^2/8$$

and

$$\max_{P\in\Omega_h} |\, Z_1 - Z_0 \,| \leqslant \min\{a^{1/2}b^{1/2}C_2, 3C_3C_1\} \| \xi_1 - \xi_2 \|_2, \tag{2.12}$$

where $C_3 = C_3(d, \alpha)$ *is a positive decreasing function of the diameter* $d = (a^2 + b^2)^{1/2}$ *such that*

$$\max_{P\in\Omega_h} h^2 \sum_{Q\in\Omega_h} G^2(P; Q) \leqslant C_3{}^2. \tag{2.13}$$

Proof. We prove the estimate in (2.11).

Let $\xi_t = \xi_2 + t(\xi_1 - \xi_2)$ with $t \in [0, 1]$; note that $\xi_0 = \xi_2$ and $\xi_1 = \xi_1$. Let $Z_t = Z(\xi_t ; W_{0t} : P)$, where $W_{0t} \in \mathscr{W}(\xi_t)$; note that $W_{00} \in \mathscr{W}(\xi_0) = \mathscr{W}(\xi_2)$.

If $t \in (0, 1]$, then, at each point of Ω_h,

$$\begin{aligned}\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t]& \\ + \{\mathscr{L}_h(W_{0t}) - \mathscr{L}_h(W_{00})\}[\xi_0] + \mathscr{L}_h(W_{0t})[\xi_t - \xi_0]&\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t]& \\ + \{\mathscr{L}_h(W_{0t}) - \mathscr{L}_h(W_{00})\}[\xi_t] + \mathscr{L}_h(W_{00})[\xi_t - \xi_0];&\end{aligned} \tag{2.15}$$

here

$$\{\mathscr{L}_h(W_{0t}) - \mathscr{L}_h(W_{00})\}[\xi_t] = (a(P, W_{0t}(P)) - a(P, W_{00}(P)))\, \xi_{tx\bar{x}} + \cdots.$$

Observe that (2.14) and (2.15) are completely interchangeable and they are a result of algebraic grouping and assume no properties of W_{0t} or ξ_t.

We will now partition Ω_h. Let $\Omega_h^{(i)}(t) = \{P: P \in \Omega_h$ and exactly i of the expressions $\xi_{tx\bar{x}}(P)$, $\xi_{txy}(P)$, $\xi_{ty\bar{y}}(P)$ are zero$\}$; note that $\xi_{tx\bar{x}}(P) = (1 - t)\, \xi_{2x\bar{x}} + t\xi_{1x\bar{x}}$, etc. For every $t \in [0, 1]$,

$$\Omega_h = \sum_{i=0}^{3} \Omega_h^{(i)}(t)$$

with $\Omega_h^{(i)}(t) \cap \Omega_h^{(j)}(t) = 0$ for $i \neq j$.

Now we partition $\Omega_h^{(1)}(t)$. Let $\Omega_h^{(1,1)}(t) = \{P: P \in \Omega_h^{(1)}(t),\ \xi_{tx\bar{x}}(P) = 0,\ |\, \xi_{txy}(P)\, \xi_{ty\bar{y}}(P)| > 0\}$,

$$\Omega_h^{(1,2)}(t) = \{P: P \in \Omega_h^{(1)}(t),\ \xi_{txy}(P) = 0,\ |\, \xi_{tx\bar{x}}(P)\, \xi_{ty\bar{y}}(P)| > 0\},$$

and $\Omega_h^{(1,3)}(t) = \{P: P \in \Omega_h^{(1)}(t),\ \xi_{ty\bar{y}}(P) = 0,\ |\, \xi_{tx\bar{x}}(P)\, \xi_{txy}(P)| > 0\}$. For each $t \in [0, 1]$,

$$\Omega_h^{(1)}(t) = \sum_{i=1}^{3} \Omega_h^{(1,i)}(t),$$

where $\Omega_h^{(1,i)}(t) \cap \Omega_h^{(1,j)}(t) = 0$ for $i \neq j$.

Now we partition $\Omega_h^{(2)}(t)$. Let $\Omega_h^{(2,1)}(t) = \{P: P \in \Omega_h^{(2)}(t), \xi_{tx\bar{x}}(P) = \xi_{txy}(P) = 0, \mid \xi_{ty\bar{y}}(P)\mid > 0\}$, $\Omega_h^{(2,2)}(t) = \{P: P \in \Omega_h^{(2)}(t), \xi_{tx\bar{x}}(P) = \xi_{ty\bar{y}}(P) = 0, \mid \xi_{txy}(P)\mid > 0\}$, and

$$\Omega_h^{(2,3)}(t) = \{P: P \in \Omega_h^{(2)}(t), \xi_{txy}(P) = \xi_{ty\bar{y}}(P) = 0, \mid \xi_{tx\bar{x}}(P)\mid > 0\}.$$

For each $t \in [0, 1]$,

$$\Omega_h^{(2)}(t) = \sum_{i=1}^{3} \Omega_h^{(2,i)}(t),$$

where $\Omega_h^{(2,i)}(t) \cap \Omega_h^{(2,j)}(t) = 0$ for $i \neq j$.

Let $P \in \Omega_h$. Then, for some i, $P \in \Omega_h^{(i)}(0)$.

If $i = 3$, the identity in (2.14) implies that

$$\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t] + \mathscr{L}_h(W_{0t})[\xi_t - \xi_0] \tag{2.16}$$

for all $t \in [0, 1]$; all the second order difference quotients of ξ_0 are zero. In this case $\mathscr{W}(\xi_t : Q) = [A, B]$.

Suppose that $i = 0$. Then $\xi_{0x\bar{x}}(P) \cdot \xi_{0xy}(P) \cdot \xi_{0y\bar{y}}(P) \neq 0$ and there exists a largest positive number $\tau_0(P)$ such that for all $t \in [0, \tau_0(P))$ we have that $\xi_{tx\bar{x}}(P) \cdot \xi_{txy}(P) \cdot \xi_{ty\bar{y}}(P) \neq 0$; this follows since ξ_t is a linear function of t. Let $\tau_0 = \min\{\tau_0(P): P \in \Omega_h^{(0)}(0)\}$. Then $\tau_0 > 0$ since $\Omega_h^{(0)}(0)$ has a finite number of elements. Therefore, for all $P \in \Omega_h^{(0)}(0)$ and for all $t \in [0, \tau_0)$, the equation in (2.14) becomes

$$\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t] + \mathscr{L}_h(W_{0t})[\xi_t - \xi_0];$$

this follows since the elements of $\mathscr{W}(\xi_t : P)$ are completely determined by the algebraic sign of the difference quotients of ξ_t evaluated at P.

Now we ask what happens at $t = \tau_0$. Here we have that, for some $P_0 \in \Omega_h^{(0)}(0)$, $\xi_{\tau_0 x\bar{x}}(P_0) \cdot \xi_{\tau_0 xy}(P_0) \cdot \xi_{\tau_0 y\bar{y}}(P_0) = 0$ and this is the first value of $t \in [0, \tau_0(P_0))$ for which this happens. The algebraic sign of each second order difference quotient of $\xi_t(P_0)$ which does not vanish at $t = \tau_0$ remains of the same algebraic sign for $t = \tau_0$ as for $t < \tau_0$. Hence the numerical value of each principal coefficient of $\mathscr{L}_h$ which corresponds to a nonvanishing second order difference quotient of $\xi_t(P_0)$ at $t = \tau_0$ is the same as for $t < \tau_0$. Hence (2.15) implies that

$$\gamma L_h[Z_0 - Z_{\tau_0}] = \gamma L_h[\xi_0 - \xi_{\tau_0}] + \mathscr{L}_h(W_{0\tau_0})[\xi_{\tau_0} - \xi_0].$$

Now suppose that $i = 1$. Then exactly one of the difference quotients of $\xi_0(P)$ is zero. Suppose that $P \in \Omega_h^{(1,1)}(0)$ with $\xi_{0xy}(P) > 0$ and $\xi_{0y\bar{y}}(P) < 0$. Then $\xi_{tx\bar{x}}(P) = t\xi_{1x\bar{x}}(P)$ for all $t \in [0, 1]$; say $\xi_{1x\bar{x}}(P) > 0$. Let $\tau_{11}(P)$ be the largest positive number in $[0, 1]$ such that $\xi_{txy}(P) > 0$ and $\xi_{ty\bar{y}}(P) < 0$

for all $t \in [0, \tau_{11}(P))$. Then $W_{0t} \in \mathscr{W}(\xi_t : P)$ implies that $a(P, W_{0t}) = \max\{a(P, X): X \in [A, B]\}$, $b(P, W_{0t}) = \max\{b(P, X): X \in [A, B]\}$, and

$$c(P, W_{0t}) = \min\{c(P, X): X \in [A, B]\} \qquad \text{for all} \quad t \in [0, \tau_{11}(P)).$$

Suppose, at $t = \tau_{11}(P)$, we have that $\xi_{\tau_{11}xy}(P) = 0$ but $\xi_{\tau_{11}y\bar{y}}(P) < 0$. Then $a(P, W_{0\tau_{11}}) = a(P, W_{00})$ and $c(P, W_{0\tau_{11}}) = c(P, W_{00})$. Therefore

$$\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t] + \mathscr{L}_h(W_{0t})[\xi_t - \xi_0] \tag{2.17}$$

for all $t \in [0, \tau_{11}(P))$ and

$$\gamma L_h[Z_0 - Z_{\tau_{11}}] = \gamma L_h[\xi_0 - \xi_{\tau_{11}}] + \mathscr{L}_h(W_{0\tau_{11}})[\xi_{\tau_{11}} - \xi_0]. \tag{2.18}$$

Let $\tau_{11} = \min\{\tau_{11}(P): P \in \Omega_h^{(1,1)}(0)\}$. Then the identifies in (2.17) and (2.18) hold for all $P \in \Omega_h^{(1,1)}(0)$ and for all $t \in [0, \tau_{11}]$.

At this point we emphasize that there is no claim as to the continuity in t of W_{0t}; in general it is discontinuous in t.

In general, to each $P \in \Omega_h^{(i,j)}(0)$, there corresponds the largest positive number $\tau_{ij}(P) \in [0, 1]$ so that for all $t \in [0, \tau_{ij}(P))$ we have $P \in \Omega_h^{(i,j)}(t)$. Proceeding as above we see that (2.17) holds for $t \in [0, \tau_{ij}(P))$ and (2.18) holds at $t = \tau_{ij}(P)$. Set $\tau_{ij} = \min\{\tau_{ij}(P): P \in \Omega_h^{(i,j)}(0)\}$.

Let $t_1 = \min\{\tau_0, \min_{i,j} \tau_{ij}\}$; clearly $t_1 > 0$ as there are only a finite number of positive quantities τ_0 and τ_{ij}. Then for all $P \in \Omega_h$

$$\gamma L_h[Z_0 - Z_t] = \gamma L_h[\xi_0 - \xi_t] + \mathscr{L}_h(W_{0t})[\xi_t - \xi_0], \qquad t \in [0, t_1),$$

and

$$\gamma L_h[Z_0 - Z_{t_1}] = \gamma L_h[\xi_0 - \xi_{t_1}] + \mathscr{L}_h(W_{0t_1})[\xi_{t_1} - \xi_0].$$

Therefore, as in (2.5), for all $t \in [0, t_1]$

$$\| Z_0 - Z_t \|_2^2 \leqslant C_2^2 t^2 \| \xi_1 - \xi_0 \|_2^2 .$$

Now we perform a similar analysis to show that there exists a $t_2 > t_1$ such that

$$\| Z_{t_1} - Z_t \|_2^2 \leqslant C_2^2 (t - t_1)^2 \| \xi_1 - \xi_0 \|_2^2 \tag{2.19}$$

for all $t \in [t_1, t_2]$. In deriving (2.19) we use the equation

$$\gamma L_h[Z_{t_1} - Z_t] = \gamma L_h[\xi_{t_1} - \xi_t] + \{\mathscr{L}_h(W_{0t}) - \mathscr{L}_h(W_{0t_1})\}[\xi_{t_1}]$$
$$+ \mathscr{L}_h(W_{0t})[\xi_t - \xi_{t_1}]$$

instead of (2.14) and we use the equation

$$\gamma L_h[Z_{t_1} - Z_t] = \gamma L_h[\xi_{t_1} - \xi_t] + \{\mathscr{L}_h(W_{0t}) - \mathscr{L}_h(W_{0t_1})\}[\xi_t] + \mathscr{L}_h(W_{0t_1})[\xi_t - \xi_{t_1}]$$

instead of (2.15).

Continuing in this manner we obtain a set $T = \{t_i : i \in I \text{ and } t_i \in [0, 1]\}$, where I is some ordered index set such that $t_i > t_j$ for $i > j$.

Now we show that I has a finite number of elements. The value of a given t_i is determined when a point P leaves some set $\Omega_h^{(i,j)}(t)$ or $\Omega^{(0)}(t)$. But for $P \in \Omega_h - \Omega_h^{(3)}$, $\Omega_h^{(3)} = \{P : P \in \Omega_h \text{ and } P \in \Omega_h^{(3)}(t) \text{ for all } t \in [0, 1]\}$, each second order difference quotient of $\xi_t(P)$ may vanish for at most one value of t; if $P \in \Omega_h^{(3)}$, then (2.16) holds for all $t \in [0, 1]$. Since Ω_h has a finite number of elements, I must have a finite number of elements. Let $I = \{0, 1,..., N + 1\}$ with $t_0 = 0$ and $t_{N+1} = 1$.

We have that

$$Z_0 - Z_1 = \sum_{i=0}^{N} (Z_{t_{i+1}} - Z_{t_i})$$

and

$$\| Z_0 - Z_1 \|_2 \leqslant \sum_{i=0}^{N} \| Z_{t_{i+1}} - Z_{t_i} \|_2 .$$

Therefore

$$\| Z_0 - Z_1 \|_2 \leqslant C_2 \left(\sum_{i=0}^{N} | t_{i+1} - t_i | \right) \| \xi_1 - \xi_2 \|_2$$

since

$$\| \xi_{t_{i+1}} - \xi_{t_i} \|_2^2 = | t_{i+1} - t_i |^2 \| \xi_1 - \xi_2 \|_2^2 .$$

Hence the inequality in (2.11) follows.

The proof of (2.12) follows from earlier results and the representation in (2.9).

At this point we wish to emphasize that if $b \equiv 0$ on $\Omega_h \times [A, B]$, then $\alpha = 0$ and the proof above would give no reference to ξ_{txy}. However, even in this case, the estimate in (2.5) remains valid.

A point $P \in \Omega_h$ is *semiplanar* (or *planar*) relative to $\xi \in \mathscr{S}_h$ if $\xi_{x\bar{x}}(P) \cdot \xi_{xy}(P) \cdot \xi_{y\bar{y}}(P) = 0$ (or $\xi_{x\bar{x}}(P) = \xi_{xy}(P) = \xi_{y\bar{y}}(P) = 0$). If $b(P, X) = 0$ for all $X \in [A, B]$, then we would exclude any reference to $\xi_{xy}(P)$ in these definitions.

Our last result may be improved if we exclude semiplanar points.

LEMMA 7. *If $\xi \in \mathscr{A}_h$ and if no point if Ω_h is semiplanar relative to ξ, then $dZ(\xi + \epsilon\eta, W_0(\xi + \epsilon\eta) : P)/d\epsilon$ exists at $\epsilon = 0$ with any $\eta \in \mathscr{A}_h$ and*

$$\gamma L_h[dZ/d\epsilon] = \gamma L_h[\eta] - \mathscr{L}_h(W_0(\xi))[\eta]$$

where $dZ/d\epsilon$ is the derivative evaluated at $\epsilon = 0$.

Proof. In the absence of semi-planar points it is clear that $W_0(\xi + \epsilon\eta : P)$ is continuous at $\epsilon = 0$; i.e., $W_0(\xi + \epsilon\eta : P) \in \mathscr{W}(\xi : P)$ for ϵ small.

Our next result will establish some essential elementary properties.

LEMMA 8. (a) *Let the hypotheses of Lemma 5 be satisfied and let $W(P)$ be an arbitrary element of $\mathscr{H}_h^0$. Then*

$$Z(U_W ; W : P) = U_W(P) \qquad \textit{and} \qquad Z(U_W ; W_0(U_W) : P) \geqslant U_W(P).$$

(b) *Let the hypotheses of Lemma 6 be satisfied. Then $Z(\xi; W_0(\xi) : P) = \xi(P) \in \mathscr{S}_h$ for all $P \in \Omega_h$ if and only if $\xi(P)$ solves* (2.1) *with discrete profile $W_0(\xi)$ and*

$$\xi(P) \geqslant U_R(P)$$

for all $P \in \Omega_h$ and for each $R \in \mathscr{H}_h^0$.

(c) *If $\xi_n \in \mathscr{S}_h$, ξ_n converges pointwise to $\xi_0 \in \mathscr{S}_h$ and $W_0(\xi_n)$ converges pointwise to V, then $Z(\xi_n ; W_0(\xi_n) : P)$ converges pointwise to $Z(\xi_0 ; V : P)$ and $V(P) \in \mathscr{W}(\xi_0 : P)$ for each $P \in \Omega_h$.*

Proof. (a) This is an immediate consequence of the representation in (2.9).

(b) Suppose that $Z(\xi; W_0(\xi) : P) = \xi(P)$ for each $P \in \Omega_h$. Then we clearly have that $\xi(P)$ solves (2.1) with discrete profile $W_0(\xi)$ and, because

$$-\mathscr{L}_h(R)[\xi] \geqslant -\mathscr{L}_h(W_0(\xi))[\xi] = -f = -\mathscr{L}_h(R)[U_R],$$

the assertion that ξ is a discrete maximal solution follows from *Lemma* 1.

Suppose that U_V is a maximal solution and that at some point of Ω_h, say P_0, $V(P_0) \notin \mathscr{W}(U_V : P_0)$. Let $Y(P) = V(P)$ for all $P \in \Omega_h - \{P_0\}$ and $Y(P_0) = W_0(U_V : P_0)$. Then

$$-\mathscr{L}_h(Y)[U_V] \leqslant -\mathscr{L}_h(V)[U_V] = -f = -\mathscr{L}_h(Y)[U_Y]$$

and $U_V \leqslant U_Y$. Therefore, either $U_V = U_Y$ for all $P \in \Omega_h$ or else $V \in \mathscr{W}(U_V : P_0)$.

(c) Suppose there exists $P_0 \in \Omega_h$ such that

$$Z(\xi_0 ; V : P_0) + \epsilon = Z(\xi_0 ; W_0(\xi_0) : P_0)$$

with $\epsilon > 0$. Let n be so large that $Z(\xi_0 ; V : P) = Z(\xi_n ; W_0(\xi_n) : P) + \epsilon_1(P)$ and $Z(\xi_0 ; W_0(\xi_0) : P) = Z(\xi_n ; W_0(\xi_0) : P) + \epsilon_2(P)$, where

$$\max\{|\epsilon_1(P)|, |\epsilon_2(P)|\} < \epsilon/2 \qquad \text{for all} \quad P \in \Omega_h .$$

Then

$$Z(\xi_n ; W_0(\xi_n) : P_0) + \epsilon_1(P_0) - \epsilon_2(P_0) + \epsilon = Z(\xi_n ; W_0(\xi_0) : P_0)$$

with $\epsilon_1(P_0) - \epsilon_2(P_0) + \epsilon > 0$. Therefore,

$$Z(\xi_n ; W_0(\xi_n) : P_0) < Z(\xi_n ; W_0(\xi_0) : P_0);$$

this is a contradiction.

Our next result will establish existence and uniqueness of discrete maximal solutions; we do not claim that a discrete maximal profile is unique.

THEOREM 1. (a) *If the condition* (I) *is satisfied, then there exists at most one discrete maximal solution.*

(b) *If the hypotheses of Lemma* 6 *are satisfied and if* $C_2 < 1$, *then there exists a discrete maximal solution and this may be obtained in an iterative manner starting with any element in* $\mathscr{S}_h$.

(c) *If the hypotheses of Lemma* 6 *are satisfied and if* $Z(\xi; W_0(\xi) : P) \in \mathscr{S}_h$ *for all* $\xi \in \mathscr{S}_h$, *then a discrete maximal solution exists.*

(d) *The existence or non-existence of a discrete maximal solution may be determined by solving* (2.1) *for a finite number of profiles in* $\mathscr{H}_h^0$ *even though that set contains an uncountably infinite number of elements.*

Proof. (a) This is immediate.

(b) Let $Z_{(0)}(P)$ be any element of $\mathscr{S}_h$. Let $Z_{(n)}(P) = Z(Z_{(n-1)} ; W_0(Z_{(n-1)}) : P) : P)$ for $n \geqslant 1$. Then the sequence $\{Z_{(n)}(P): n = 1, 2,...\}$ converges to one and only one element $\xi \in \mathscr{S}_h$. Let $V(P)$ be any accumulation point of $\{W_0(Z_{(n)} : P): n = 1, 2,...\}$; the existence of $V(P)$ follows from the *Heine–Borel Theorem* and the fact that Ω_h has finite cardinality. Then for some subsequence, denoted by n', we have that $| W_0(Z_{(n')} : P) - V(P)|$ converges to zero as n' goes to infinity for all $P \in \Omega_h$. Now apply *Lemma* 8 to conclude that $\xi(P) = U_V(P)$ is the discrete maximal solution and $V(P)$ is a discrete maximal profile.

(c) This follows from the *Brouwer Fixed Point Theorem* and *Lemma* 8.

(d) Let $\mathscr{H}_h^{(1)} = \{V(P): V \in \mathscr{H}_h^0$ and, for each $P \in \Omega_h$, $a(P, V(P))$, $b(P, V(P))$ and $c(P, V(P))$ corresponds to one of the 8 possible combinations in (III)$\}$. Then the cardinality of $\mathscr{H}_h^{(1)}$ is at most 8^K, K the number of mesh points in Ω_h, and, by *Lemma* 8(b), any discrete maximal solution must have its discrete profile in $\mathscr{H}_h^{(1)}$. Hence one computes each element in $\{U_V: V \in \mathscr{H}_h^{(1)}\}$, this has cardinality of at most 8^K, and one tests this finite set to determine if there exists a $V_0 \in \mathscr{H}_h^{(1)}$ such that

$$U_{V_0}(P) \geqslant U_V(P)$$

for all $P \in \Omega_h$ and for each $V \in \mathscr{H}_h^{(1)}$.

The condition $C_2 < 1$ is satisfied whenever $|\lambda_1 - \lambda_0|$ is sufficiently small; a case often met in applied problems. If $b(P, X) = 0$ on $\Omega \times [A, B]$, $\alpha = 0$, $\lambda_1 = 1$, and $\gamma = \lambda_1$, then $C_2 < 1$ whenever $|\lambda_1 - \lambda_0| < 1/\sqrt{6}$; the requirement $\lambda_1 = 1$ leads to no loss in generality.

Now we will make some general comments on discrete maximal profiles and on their computation whenever the condition $C_2 < 1$ is satisfied.

Let $P_0 \in \Omega_h$ be such that the algebraic sign of each of the terms $Z_{(n)x\bar{x}}(P_0)$, $Z_{(n)xy}(P_0)$, and $Z_{(n)y\bar{y}}(P_0)$ remains unchanged for all "sufficiently large" n. Then the numerical value of each of the terms $a(P_0, W_0(Z_{(n)} : P_0))$, $b(P_0, W_0(Z_{(n)} : P_0))$, and $c(P_0, W_0(Z_{(n)} : P_0))$ remains constant for all "sufficiently large" n although the value of $W_0(Z_{(n)} : P_0)$ which we selected in $\mathscr{W}(Z_{(n)} : P_0)$ *may* fluctuate wildly with n. Hence, in this case, a value of $V(P_0)$—a discrete maximal profile—may be determined in a finite number of computations.

By "sufficiently large" n we mean that there exists an integer N such that the difference quotients of $Z_{(N)}(P_0)$ are large in comparison to $|(Z_{(m)} - Z_{(N)})/h^2|$ for all $m > N$; note that $Z_{(n)}$ converges as a geometric series.

This analysis will be radically altered if some second order difference quotient of $Z_{(n)}(P)$ converges to zero and algebraic signs are not maintained as n increases. For example, suppose that $Z_{(n)xy}(P_1)$ and $Z_{(n)y\bar{y}}(P_1)$ maintain their algebraic sign for large n but $Z_{(n)x\bar{x}}(P_1) \cdot Z_{(n+1)x\bar{x}}(P_1) < 0$ for all large n and $Z_{(n)x\bar{x}}(P_1) \to 0$. Then $\{W_0(Z_{(n)} : P): n = 0, 1, ...\}$ must have at least two accumulation points unless $a(P_1, X)$ is a constant in X. One such point will be a maximum of $a(P_1, X)$ and another such accumulation point will be a minimum of $a(P_1, X)$. As there is no *a priori* way of excluding the occurrence of such points P_1 (except for planar points–require $|f(P)| > 0$ over Ω) we shall modify our definition of $W_0(Z_{(n)} : P)$ in such a way that, for each $P \in \Omega_h$, the new sequence of functions converges, as $n \to \infty$, to a unique limit and this limit is a discrete maximal profile.

Let $\hat{\Omega}_h^{(i)} = \{P: P \in \Omega_h$ and exactly i of the difference quotients of $Z_{(n)}(P)$ do not maintain their algebraic sign as $n \to \infty\}$; here $i = 0, 1, 2$, or 3. If for example $Z_{(n)x\bar{x}}(P_0) \to 0$ as $n \to \infty$ in an oscillatory manner (i.e., for every n there is an $n' > n$ such that $Z_{(n)x\bar{x}}(P_0) \cdot Z_{(n')x\bar{x}}(P_0) < 0$) but

$Z_{(n)xy}(P_0) > 0$ and $Z_{(n)y\bar{y}}(P_0) < 0$ for all large n, then $P_0 \in \Omega_h^{(1)}$. It is clear that $\Omega_h = \sum_{i=0}^{3} \hat{\Omega}_h^{(i)}$ with $\hat{\Omega}_h^{(i)} \cap \hat{\Omega}_h^{(j)} = 0$ for $i \neq j$.

Now we are prepared to modify our definition of $W_0(Z_{(n)} : P)$; the modified functions will be called $W_1(Z_{(n)} : P)$.

If $P \in \hat{\Omega}_h^{(0)}$, we define $W_1(Z_{(n)} : P) = W_0(Z_{(n)} : P)$. If $P \in \hat{\Omega}_h^{(3)}$, then set $W_1(Z_n : P) = A$. If $P \in \hat{\Omega}_h^{(1)} + \hat{\Omega}_h^{(3)}$, then $W_1(Z_{(n)} : P)$ is a minimizing or maximizing element of those coefficients in (2.9) for which the corresponding difference quotients do not tend to zero; e.g., if $Z_{(n)x\bar{x}}(P_0) \to 0$ as $n \to \infty$ in an oscillatory manner but $Z_{(n)xy}(P_0) > 0$ and $Z_{(n)y\bar{y}}(P_0) < 0$ for all sufficiently large n, then $W_1(Z_{(n)} : P)$ is a minimizing value of $c(P, X)$ and a maximizing value for $b(P, X)$. Basically $W_1(Z_{(n)} : P)$ ignores those principal coefficients for which the corresponding difference quotients of $Z_{(n)}$ tend to zero.

Let $M_{(n)}(P)$ be the solution to the problem

$$\gamma L_h[M_{(n)}(P)] = \gamma L_h[Z_{(n)}(P)] - \mathscr{L}_h(W_1(Z_{(n)} : P))[Z_{(n)}(P)] + f(P), \quad P \in \Omega_h,$$
$$M_{(n)}(P) = 0, \qquad P \in \partial\Omega_h .$$

Then our next result is easily proved.

COROLLARY 1. *If the hypotheses of Theorem* 1(b) *are satisfied, then* $M_{(n)}$ *and* $Z_{(n)}$ *converge to the same element and the* $W_1(Z_{(n)} : P)$ *converges to an element* $W_1(P)$ *which is a discrete maximal profile.*

The restriction that Ω be a rectangular region is a result of the difference problem in (2.1) and the estimate in (2.5). Using a different difference formulation we may extend the results of this section to more general regions Ω (e.g., Ω a disc) using results in [3].

3. THE DIFFERENTIAL EQUATION AND CONVERGENCE WITH (III)

In this section we will show that—under suitable hypotheses—the problem in (1.2) has a solution and that solutions of the discrete problem converge to it.

Let $z(\xi; w : P) \in H_{2,0}^2(\Omega)$ be the solution of the equation

$$\gamma L[z] = \gamma L[\xi] - \mathscr{L}(w)[\xi] + f, \tag{3.1}$$

where $\gamma > 0$, $w \in \mathscr{H}^0$, $\xi \in H_{2,0}^2(\Omega)$, and for any $v \in H_{2,0}^2(\Omega)$

$$L[v] = v_{,xx} + \alpha v_{,xy} + v_{,yy}$$

with $\alpha \in [0, 1]$.

LEMMA 9. (a) *Let the conditions in* (I) *and* (II) *be satisfied. Let* $w \in \mathscr{H}^0$ *and let* $v \in H^2_{2,0}(\Omega)$ *solve the equation* $\mathscr{L}(w)[v] = f \in L_p(\Omega)$ *with* $p > 2$. *If* $f \geqslant 0$ *a.e. over* Ω, *then* $v \leqslant 0$ *over* Ω.

(b) *If the conditions in* (I), (II), *and* (III) *are satisfied and if* $\xi \in H^2_{2,0}(\Omega)$, *then there exists* $w_0(\xi : P) \in \mathscr{H}^0$ *such that*

$$z(\xi; w_0(\xi) : P) \geqslant z(\xi; w : P)$$

for each $P \in \Omega$ *and for all* $w \in \mathscr{H}^0$.

Proof. (a) This is an immediate consequence of results in Bers and Nirenberg [1, pp. 154–156] and Talenti [9].

(b) The proof here proceeds as in Section 2 once we observe that $u \in H^2_{2,0}(\Omega)$ implies that second order derivatives are finite almost everywhere and hence $w_0(\xi : P)$ is defined for a particular representative of ξ.

Let h be a positive number such that the set Ω_h is not empty and $\partial\Omega_h \subset \partial\Omega$. Let h_n, n ranges over the nonnegative integers, be a monotone decreasing sequence of real positive numbers tending to zero, with $h_0 = h$, such that $\Omega_{h_n} \subset \Omega_{h_m}$ and $\partial\Omega_{h_n} \subset \partial\Omega_{h_m}$ for each n and for all $m \geqslant n$; for brevity we let $\Omega_n = \Omega_{h_n}$.

Let $\Omega_n' = \Omega_n + s_{(n)3} + s_{(n)4}$, where $s_{(n)j} = s_{(h_n)j}$, and let $P_0 = (x_0, y_0) \in \Omega_h'$. Let $\mathscr{N}_n(P_0) = \{(x, y): x_0 \leqslant x < x_0 + h_n, y_0 \leqslant y < y_0 + h_n\}$ and $\mathscr{P}_n = \bigcup \{\mathscr{N}_n(P): P \in \Omega_n'\}$. Let n be given. Then for each and every $P \in \Omega$ there exists one and only one $P_0 \in \Omega_n'$ such that $P \in \mathscr{N}_n(P_0)$; in particular, $\Omega + s_3 + s_4 = \mathscr{P}_n$.

Let $\xi \in C_0^\infty(\Omega)$. Then we may reflect ξ across the $\partial\Omega$ so that the extended function is infinitely differentiable over $\{(x, y): -h_0 \leqslant x \leqslant a + h_0, -h_0 \leqslant y \leqslant b + h_0\}$. Let $f \in C(\bar{\Omega})$. For each $P \in \Omega_n'$ and for each $\mu \in \mathscr{H}^0_{h_n}$ we let

$$F_n(\xi; \mu : P) = \gamma L_{(n)}[\xi(P)] - \mathscr{L}_{(n)}(\mu(P))[\xi(P)] + f(P),$$

where $L_{(n)} = L_{h_n}$ and $\mathscr{L}_{(n)} = \mathscr{L}_{h_n}$ are defined as in Section 2. Now let

$$\mathscr{F}_n(\xi; \mu : P) = F_n(\xi; \mu : P_0),$$

where P is an arbitrary element of Ω and P_0 is the unique element of Ω_h' such that $P \in \mathscr{N}_n(P_0)$.

LEMMA 10. (a) *Let the conditions in* (I), (II), *and* (III) *be satisfied with* $f \in C(\bar{\Omega})$. *For each* $\mu \in \mathscr{H}^0_{h_n}$ *and for each* $\xi \in C_0^\infty(\Omega)$ *let* $z_n = z_n(\xi; \mu : P) \in H^2_{2,0}(\Omega)$ *solve the equation*

$$\gamma L[z_n] = \mathscr{F}_n(\xi; \mu : P) \tag{3.2}$$

for $P \in \Omega$. *Then there exists an* $w_{0n}(\xi : P) \in \mathscr{H}^0_{h_n}$ *such that*

$$z_n(\xi; w_{0n} : P) \geqslant z_n(\xi; \mu : P)$$

for all $P \in \Omega$ *and for each* $\mu \in \mathscr{H}^0_{h_n}$.

(b) *If the hypotheses in* (a) *are satisfied and if* $\xi_1, \xi_2 \in C_0^\infty(\Omega)$, *then*

$$| z_{1n} - z_{2n} |_2^2 \leqslant C_2^2 \| \xi_1 - \xi_2 \|_2^2, \tag{3.3}$$

where

$$| z |^2 = \int_\Omega \{z^2_{,xx} + 2z^2_{,xy} + z^2_{,yy}\} \, dx$$

and $z_{in} = z_n(\xi_i ; w_{0n}(\xi_i) : P)$ *for* $i = 1, 2$.

(c) *Let the conditions in* (I), (II), *and* (III) *be satisfied with* $f \in C(\bar{\Omega})$. *If* $\xi_i \in C_0^\infty(\Omega)$ *for* $i = 1, 2$, *then* z_{in} *converges strongly to* z_i (z_i *is given in* Lemma 9(b)) *in* $H^2_{2,0}(\Omega)$ *as* $n \to \infty$ *and*

$$| z_1 - z_2 |_2^2 \leqslant C_2^2 | \xi_1 - \xi_2 |_2^2. \tag{3.4}$$

Moreover, for $\xi_1, \xi_2 \in H^2_{2,0}(\Omega)$, *we have that*

$$| z_1 - z_2 |_2^2 \leqslant C_2^2 | \xi_1 - \xi_2 |_2^2. \tag{3.5}$$

(d) *Each of the above results holds if* $f \in L_p(\Omega)$ *with* $p > 2$.

Proof. (a) and (b) are proved as in Section 2; note that

$$\int_\Omega \mathscr{F}_n^2(\xi : P) \, dP = h_n^2 \sum_{\Omega_n'} \mathscr{F}_n^2(\xi : P).$$

(c) If $\xi \in C_0^\infty(\Omega)$, then second order difference quotients of ξ converge strongly in $L_2(\Omega)$ to corresponding second order derivatives of ξ. Thus there exists a subsequence of the n so that this convergence is convergence a.e. Hence $\gamma L_{(n)}[\xi] - \mathscr{L}_{(n)}(w_{0n}(\xi))[\xi] + f$ converges almost everywhere to $\gamma L[\xi] - \mathscr{L}(w_0(\xi))[\xi] + f$. This occurs even if $w_{0n}(\xi : P)$ does not converge to $w_0(\xi; P)$; see the discussion preceding *Corollary* 1. Therefore (3.4) is a consequence of (3.3).

If $\xi_i \in H^2_{2,0}(\Omega)$, $i = 1, 2$, then there exists sequences from $C_0^\infty(\Omega)$ which converge strongly in $H^2_{2,0}(\Omega)$ to ξ_i and the analysis of the preceding paragraph remains valid. Therefore (3.5) is a consequence of (3.4).

(d) If $f \in L_p$, then we may find a sequence of elements from $C(\bar{\Omega})$ which converge strongly to f in $L_p(\Omega)$. The associated solutions of (3.1) converge strongly in $H^2_{2,0}(\Omega)$ and our analysis extends to this more general case.

Let

$$\mathscr{S} = \{\xi : \xi \in H^2_{2,0}(\Omega) \quad \text{and} \quad |\xi|_2^2 \leqslant 2\lambda_1^2 |f|^2/\lambda_0^4\},$$

where

$$|f|^2 = \int_\Omega f^2 \, dP.$$

Our next result establishes existence and uniqueness for the problem in (1.2); its proof follows closely that of Section 2.

THEOREM 2. (a) *Let the conditions* (I), (II), *and* (III) *be satisfied. If there exists* $\xi \in \mathscr{S}$ *such that* $z(\xi; w_0(\xi) : P) = \xi(P)$ *for* $P \in \Omega$, *then*

$$\xi(P) \geqslant u_v(P)$$

for all $P \in \Omega$ *and for each* $v \in \mathscr{H}^0$.

(b) *If* (I), (II), *and* (III) *are satisfied and if* $C_2 < 1$, *then the problem in* (1.2) *has a unique maximal solution.*

Now we will prove the convergence of the discrete maximal solutions to the solution of (1.2). The basic idea of the proof is that step functions are dense in $\mathscr{H}^0$ and solutions of (1.1) converge strongly in $H^2_{2,0}(\Omega)$ whenever the coefficients converge a.e.

Our proof of this result will be greatly simplified if we require that (I′) be satisfied; i.e., (I) holds and, for all $(P, X) \in \bar{\Omega} \times E^1$, $a(P, X) = a_1(P) + a_2(X)$, $b(P, X) = b_1(P) + b_2(X)$, $c(P, X) = c_1(P) + c_2(X)$.

Let n be given and let $\mathscr{E}_n = \{v(P) : v \in L_\infty(\Omega), v$ is constant on each $\mathscr{N}_n(P)$ for each $P \in \Omega_n'\}$; each element of this set is a step function.

We will now establish sufficient conditions that the problem in (1.2) has a solution over $\mathscr{E}_n' \equiv \mathscr{E}_n \cap \mathscr{H}^0$ for each n.

THEOREM 3. *If the conditions in* (I′), (II), *and* (III) *are satisfied with* $C_2 < 1$, *then, for each* $n \geqslant 0$, *there exists an* $v_n \in \mathscr{E}_n'$ *such that*

$$u_{v_n}(P) \geqslant u_v(P) \tag{3.6}$$

for every $P \in \Omega$ *and for each* $v \in \mathscr{E}_n'$; *this maximal solution is unique.*

Proof. For every $v \in \mathscr{E}_n'$ and for every $\xi \in \mathscr{S}$ we define the function $\tilde{z}(\xi; v : P) \in H^2_{2,0}(\Omega)$ such that $\gamma L[\tilde{z}] = \gamma L[\xi] - \mathscr{L}(v)[\xi] + f$. We may write

$$\begin{aligned}\tilde{z}(\xi; v : P) &= -(1/\gamma) \int_\Omega \mathscr{G}(P; Q)\{\gamma L[\xi(Q)] - \mathscr{L}(v(Q))[\xi(Q)] + f(Q)\} \, dQ \\ &= -(1/\gamma) \sum_{R \in \Omega_n'} \int_{\mathscr{N}_n(R)} \mathscr{G}(P; Q)\{\gamma L[\xi] - \mathscr{L}(v)[\xi] + f\} \, dQ;\end{aligned}$$

here $\mathscr{G}(P; Q)$ is the Green's function for the operator L over Ω. Let $w_0(\xi : R) \in [A, B]$ be such that

$$-\int_{\mathscr{N}_n(R)} \mathscr{G}(P; Q)\{\gamma L[\xi(Q)] - \mathscr{L}(w_0(\xi : R))[\xi(Q)] + f(Q)\} \, dQ$$

$$\geqslant -\int_{\mathscr{N}_n(R)} \mathscr{G}(P; Q)\{\gamma L[\xi(Q)] - \mathscr{L}(v(R))[\xi(Q)] + f(Q)\} \tag{3.7}$$

for all $v(R) \in [A, B]$ and set $w_0(\xi : Q) = w_0(\xi : R)$ for all $Q \in \mathscr{N}_n(R)$. A slight modification of the methods of Section 2 will allow us to prove that

$$| \tilde{z}(\xi_1 ; w_0(\xi_1) : P) - \tilde{z}(\xi_2 ; w_0(\xi_2) : P)|_2^2 \leqslant C_2^2 \, | \, \xi_1 - \xi_2 \, |_2^2 .$$

The result now follows.

Let n be fixed and let $m \geqslant n$. For each $R \in \Omega_n'$ let

$$\mathscr{R}_m(R) = \mathscr{N}_n(R) \cap \Omega_m .$$

Let

$$\tilde{Z}(\xi; V : P) = -(1/\gamma) \sum_{R \in \Omega_h'} h_m^2 \sum_{Q \in \mathscr{R}_m(R)} G(P; Q)\{L_{(m)}[\xi] - \mathscr{L}_{(m)}(V)[\xi] + f\} \tag{3.8}$$

for each $\xi \in \mathscr{S}_{h_m}$ and $V \in \mathscr{E}_n'$. We define $W_0(\xi)$ as in (3.7).

The next result, which may be proved as in Section 2, asserts that there exists a discrete maximal solution over $\mathscr{E}_n'$; note that when $n = m$ we have the results in *Theorem* 1.

THEOREM 4. *Let the conditions in* (I′), (II), *and* (III) *be satisfied with* $C_2 < 1$. *Then, for each* $m > 0$, *there exists a unique* $\xi \in \mathscr{S}_{h_m}$ *such that* $\tilde{Z}(\xi; W_0(\xi) : P) = \xi(P)$ *and*

$$\xi(P) \geqslant U_V(P) \tag{3.9}$$

for each $P \in \Omega_m$ *and for each* $V \in \mathscr{E}_n'$.

We will now prove that solutions of (3.9) converge to solutions of (3.6) as m goes to infinity; here n is fixed but arbitrary.

Let $\alpha = 0$ (note that this affects the value of C_2), let $G_m(P; Q)$ be the discrete Green's function for $L_{(m)}$ over Ω_m, and let $\mathscr{G}(P; Q)$ be the Green's function for L over Ω. We will now prove that

$$\lim_{m \to \infty} h_m^2 \sum_{P \in \Omega_m} G_m^2(P; Q) = \int_\Omega \mathscr{G}^2(P; Q) \, dP,$$

when $\Omega = (0, 1) \times (0, 1)$.

From references in [2, pp. 314–318] we know that

$$| G_m(P; Q) - \mathscr{G}(P; Q)| \leqslant (2.15)\, h^2/\delta_{PQ}^2 ,$$

where δ_{PQ} is the distance from P to Q and $(P, Q) \in \Omega \times \Omega$. Let

$$K_m(Q) = \{P : P \in \Omega_m{}', \delta_{PQ} > h_m^{1/2}\}.$$

Then, for $P_0 \in K_m(Q)$,

$$\int_{\mathscr{N}_m(P_0)} \mathscr{G}^2(P; Q)\, dP = \int_{\mathscr{N}_m(P_0)} G_m{}^2(P; Q)\, dP + O(h_m^{1/2}) \tag{3.10}$$

and, for every $P \in \mathscr{N}_m(P_0)$,

$$G_m(P; Q) - G_m(P_0 ; Q) = \int_{P_0}^{P} G_{m(1)}(R; Q)\, dR, \tag{3.11}$$

where the integration in (3.11) is along lines connecting P_0 to P with no line segment intersecting $K_m(Q)$ and $G_{m(1)}(R; Q)$ denotes a first order directional derivative of G_m with respect to R. It is easy to prove—with a modification of the methods in [5]—that there exists a constant $H > 0$, which is independent of h_m, such that

$$| G_{m(1)}(R; Q)| \leqslant H/\delta_{RQ} .$$

For $R \in K_m(Q)$, we have $1/\delta_{RQ} \leqslant 1/h_m^{1/2}$. Also for $P \in \mathscr{N}_m(P_0)$ we have $| P - P_0 | \leqslant h_m \sqrt{2}$. Hence

$$G_m(P; Q) = G_m(P_0 ; Q) + O(h_m^{1/2}).$$

Therefore,

$$G_m{}^2(P; Q) = G_m{}^2(P_0 ; Q) + \theta(h_m) \tag{3.12}$$

where $\theta(h_m) \to 0$ uniformly in P_0 as $m \to \infty$; e.g., see [1, p. 585] of [5].

Combining (3.10) and (3.12) gives

$$\int_{\mathscr{N}_m(P_0)} G_m{}^2(P; Q)\, dP = h_m{}^2 G_m(P_0 ; Q) + \theta(h_m)\, h_m{}^2.$$

If $K_m{}'(Q) = \{P: P \in \Omega, \delta_{PQ} \leqslant h_m^{1/2}\}$ and $\Omega_m(Q) = \Omega_m - K_m(Q)$,

$$h_m{}^2 \sum_{\Omega_m} G_m{}^2(P; Q) - \int_\Omega \mathscr{G}^2(P; Q)\, dP \to 0$$

as $m \to \infty$ since

$$\int_{K_m{}'(Q)} \mathscr{G}^2(P; Q)\, dP \to 0, \qquad h_m{}^2 \sum_{\Omega_m(Q)} G_m{}^2(P; Q) \to 0.$$

Therefore, we have proved that $G_m(P; Q)$ converges strongly in $L_2(\Omega)$ to $\mathscr{G}(P; Q)$. The above result holds with minor modification for general rectangular regions Ω.

Let $U_{(m)}(P)$ be the discrete maximal solution in (3.9) and let $W_{0(m)}(P)$ be a discrete maximal profile; we still have $\alpha = 0$.

From Stummel [7] there exists a subsequence of the m, say m', such that for every bounded and continuous a.e. function ϕ_i $(i = 1, 2, 3)$ which has compact support on the plane there exists $u_0 \in H^2_{2,0}(\Omega)$ such that each of the terms

$$h^2_{m'} \sum_{\Omega_{m'}} \phi_1 U_{(m')x\bar{x}}\,, \qquad h^2_{m'} \sum_{\Omega_{m'}} \phi_2 U_{(m')xy}\,, \qquad h^2_{m'} \sum_{\Omega_{m'}} \phi_3 U_{(m')y\bar{y}} \tag{3.13}$$

converges to the respective term

$$\int_\Omega \phi_1 u_{0,xx}\, dP, \qquad \int_\Omega \phi_2 u_{0,xy}\, dP, \qquad \int_\Omega \phi_3 u_{0,yy}\, dP.$$

Using the *Heine–Borel Theorem* we know that there exists an $w_0 \in \mathscr{E}_n'$ such that $W_{0(m'')}$ converges a.e. to w_0 for some further subsequence, say m'', of the m'. Note that over each $\mathscr{N}_n(P)$ the function $W_{0(m'')}$ must be constant for all sufficiently large m''.

In (3.13) let $\phi_1 = a(P, W_{0(m'')})\, G_k(P; Q)$, $\phi_2 = 2b(P, W_{0(m'')})\, G_k(P; Q)$ and $\phi_3 = c(P, W_{0(m'')})\, G_k(P; Q)$ for some fixed k. Since the $G_k(P; Q)$ converge strongly to $\mathscr{G}(P; Q)$, $| h_m{}^2 \sum G_m{}^2 - h_k \sum G_k{}^2 | < \epsilon$ for any $\epsilon > 0$ and for all $m > k$ and for k sufficiently large. Therefore, for any $\epsilon > 0$ there exists k_0 such that for all $k > k_0$ and for all $m \geqslant k$, we have

$$\begin{aligned} U_{(m)} &= h_m \sum_{\Omega_m} G_m J_m = h_m \sum_{\Omega_m} G_k J_m + h_m \sum_{\Omega_m} (G_m - G_k)\, J_m \\ &= \int_\Omega G_k J\, dQ + O(\epsilon) = \int_\Omega \mathscr{G} J\, dQ + O(\epsilon), \end{aligned} \tag{3.14}$$

where

$$J_m = L_{(m)}[U_{(m)}] - \mathscr{L}_{(m)}(W_{0(m)})[U_{(m)}] + f$$

and

$$J = L[u_0] - \mathscr{L}(w_0)[u_0] + f.$$

It is not difficult to see—using (3.13)—that u_0 solves the Eq. (1.1) with profile w_0. Therefore (3.14) proves the pointwise convergence of $U_{(m)}$ to u_0. In fact we have proved that, for each $v \in \mathscr{E}_n'$, the solutions U_v of the difference equation converge to the solution u_v of the differential equation.

Suppose $u_0 \in H^2_{2,0}(\Omega)$ is not maximal. Then there exists $v \in \mathscr{E}_n'$ and $P_0 \in \Omega$ such that

$$u_v(P_0) > u_0(P_0);$$

in fact, this inequality is valid in some neighborhood of P_0. Therefore, if h_m is sufficiently small, this contradicts the fact that $U_{(m)}$ is maximal.

We have proved the following result.

THEOREM 5. *If the hypotheses of* Theorem 4 *are satisfied with* $\alpha = 0$ *and* $f \in C(\bar{\Omega})$, *then the discrete maximal solutions of* (3.9) *converge* (*weakly in* $H^2_{2,0}(\Omega)$ *and pointwise*) *to the maximal solution in* (3.6).

Let w_0 be a maximal profile of *Theorem* 2 and let w_0 be a particular element from the equivalence class determined by w_0. Let $w_n \in \mathscr{E}_n'$ be such that for each $P_0 \in \Omega_n'$ and for all $P \in \mathscr{N}_n(P_0)$ we have $w_n(P) = w_0(P^*)$ with P^* an arbitrary but fixed element of $\mathscr{N}_n(P_0)$. Then w_n converges a.e. to w_0 and u_{w_n} converges strongly in $H^2_{2,0}(\Omega)$ (hence uniformly) to u_{w_0} as $n \to \infty$. Therefore for any $\epsilon > 0$ there is a sufficiently large n such that

$$0 \leqslant u_{w_0} - u_{w_n} < \epsilon$$

for all $P \in \Omega$. Hence,

$$0 \leqslant u_{w_0} - u_{v_n} + u_{v_n} - u_{w_n} < \epsilon$$

with u_{v_n} a maximal solution over $\mathscr{E}_n'$, $u_{w_0} - u_{v_n} \geqslant 0$, and $u_{v_n} - u_{w_n} \geqslant 0$. Thus solutions of (3.6) come arbitrarily close to maximal solutions over $\mathscr{H}^0$.

We have proved our final result.

THEOREM 6. *If the hypotheses of* Theorem 4 *are satisfied with* $\alpha = 0$ *and* $f \in C(\bar{\Omega})$, *then the discrete maximal solutions over* $\mathscr{H}_h^0$ *converge pointwise to maximal solutions over* $\mathscr{H}^0$.

4. THE DIFFERENCE EQUATION WITH (IV)

In this section we will indicate how the methods of Section 2 may be modified when the principal coefficients in (2.1) satisfy the conditions in (I), (II), and (IV).

We assume that $b(P, X) = 0$ for all $(P, X) \in \Omega \times E^1$; this will simplify our computations without an intrinsic restriction of our results. We assume also that Ω is a rectangular domain.

Let $(\alpha, \gamma) \in E^2$ and define

$$F(\alpha, \gamma : X) = a(P, X)\alpha + c(P, X)\gamma \tag{4.1}$$

for $X \in E^1$.

LEMMA 11. *If the conditions* (I) *and* (II) *are satisfied, then for each* $(\alpha, \gamma) \in E^2$ *there exists* $W_0(\alpha, \gamma) \in [A, B]$ *such that*

$$F(\alpha, \gamma : W_0(\alpha, \gamma)) = \max_{X \in E^1} F(\alpha, \gamma : X). \tag{4.2}$$

Proof. This follows the reasoning of *Lemma* 3 and *Lemma* 5.
In the next result we show that (IV) implies the regularity of $W_0(\alpha, \gamma)$.

LEMMA 12. *If the conditions* (I), (II), *and* (IV) *are satisfied, then* $W_0(\alpha, \gamma) \in C^1(E^2 - \{0, 0\})$.

Proof. Let $\alpha \cdot \gamma \neq 0$, $W_0 = W_0(\alpha, \gamma)$, and $W_{0\epsilon} = W_0(\alpha + \epsilon, \gamma)$. Then, by (IV)(ii),

$$0 = \partial F(\alpha, \gamma : W_0)/\partial X = a'(P, W_0)\alpha + c'(P, W_0)\gamma \tag{4.3}$$

and

$$0 = \partial F(\alpha + \epsilon, \gamma : W_{0\epsilon})/\partial X = a'(P, W_{0\epsilon})(\alpha + \epsilon) + c'(P, W_{0\epsilon})\gamma. \tag{4.4}$$

From condition (IV)(i) we know that not both $a'(P, W_0)$ and $c'(P, W_0)$ may vanish. Let us assume that $a'(P, W_0) \cdot c'(P, W_0) \neq 0$. Note that if $a'(P, W_0) \cdot c'(P, W_0) = 0$, then $\alpha \cdot \gamma = 0$; this contradicts our assumption that $\alpha \cdot \gamma \neq 0$.

From (4.3),

$$\gamma = -\alpha a'(P, W_0)/c'(P, W_0). \tag{4.5}$$

Substituting (4.5) into (4.4) gives

$$a'(P, W_{0\epsilon})\, c'(P, W_0)(\alpha + \epsilon) - c'(P, W_{0\epsilon})\, a'(P, W_0)\alpha = 0. \tag{4.6}$$

We rewrite (4.6)—using *Taylor's Theorem*—as

$$\alpha\{W_{0\epsilon} - W_0\}\, K(P) = -\epsilon c'(P, W_0)\, a'(P, W_{0\epsilon}), \tag{4.7}$$

where

$$\begin{aligned} K(P) = \int_0^1 \{&c'(P, W_0)\, a''(P, W_0 + t(W_{0\epsilon} - W_0)) \\ &- a'(P, W_0)\, c''(P, W_0 + t(W_{0\epsilon} - W_0))\}\, dt. \end{aligned} \tag{4.8}$$

Hence, for some $t_0 \in [0, 1]$,

$$\begin{aligned} K(P) = {} & c'(P, W_0)\, a''(P, W_0 + t_0(W_{0\epsilon} - W_0)) \\ & - a'(P, W_0)\, c''(P, W_0 + t_0(W_{0\epsilon} - W_0)). \end{aligned}$$

From (4.7) we claim that $W_{0\epsilon}$ is continuous in ϵ at $\epsilon = 0$.

To see this we observe that the right-hand side of (4.7) tends to zero independent of the behavior of $W_{0\epsilon}$. Clearly, if $K(P)$ is bounded away from zero as $\epsilon \to 0$ our result is established. Assume that $W_{0\epsilon}$ is not continuous at $\epsilon = 0$. Then for some subsequence of ϵ we may assume that the algebraic sign of $W_{0\epsilon} - W_0$ is constant and, for some fixed $\mu > 0$, $| W_{0\epsilon} - W_0 | \to \mu > 0$ as $\epsilon \to 0$. Now the condition (IV)(vi) implies a contradiction to the assertion that $| W_{0\epsilon} - W_0 | \to \mu > 0$. Therefore, $W_0(\alpha + \epsilon, \gamma) \to W_0(\alpha, \gamma)$ as $\epsilon \to 0$. By the same methods we have that $W_0(\alpha, \gamma + \eta) \to W_0(\alpha, \gamma)$ as $\eta \to 0$.

Now we prove that $W_0(\alpha + \epsilon, \gamma + \eta) \to W_0(\alpha, \gamma)$ as $(\epsilon, \eta) \to (0, 0)$. As in (4.3) we have that $a_1'(\alpha + \epsilon) + c_1'(\gamma + \eta) = 0 = a'\alpha + c'\gamma$, where $a_1' = a'(P, W_0(\alpha + \epsilon, \gamma + \eta))$, $a' = a'(P, W_0(\alpha, \eta))$, etc. Hence $c_1'a' - a_1'c'$ approaches zero as $(\epsilon, \eta) \to (0, 0)$. By (iii) of (IV) all accumulation points of $W_0(\alpha + \epsilon, \gamma + \eta)$ are equal; note that $a_1' \cdot c_1' \neq 0$ for small (ϵ, η) since $(\alpha + \epsilon, \gamma + \eta)$ is in the same quadrant as (α, γ).

From (4.7) and the continuity of $W_{0\epsilon}$ we conclude that

$$\frac{\partial W_0(\alpha, \gamma)}{\partial \alpha} = - \frac{c'(P, W_0)\, a'(P, W_0)}{\alpha L(P, W_0)}, \tag{4.9}$$

where

$$L(P, W_0) = c'(P, W_0)\, a''(P, W_0) - a'(P, W_0)\, c''(P, W_0)$$

with $| L(P, W_0)| > 0$.

A similar analysis will show that

$$\frac{\partial W_0(\alpha, \gamma)}{\partial \alpha} = \frac{\{a'(P, W_0)\}^2}{\gamma L(P, W_0)}$$

and

$$\frac{\partial W_0(\alpha, \gamma)}{\partial \gamma} = \frac{c'(P, W_0)\, a'(P, W_0)}{\gamma L(P, W_0)} = \frac{-\{c'(P, W_0)\}^2}{\alpha L(P, W_0)}. \tag{4.10}$$

Consider the case that $\alpha \cdot \gamma = 0$ but $\alpha^2 + \gamma^2 > 0$; suppose $\alpha = 0$. Then, by (IV)(ii) and (4.3), $c'(P, W_0) = 0$ and the equation in (4.4) becomes

$$a'(P, W_{0\epsilon})\epsilon + c'(P, W_{0\epsilon})\gamma = 0.$$

Hence

$$a'(P, W_0)[c'(P, W_{0\epsilon}) - c'(P, W_0)]\gamma = -\epsilon a'(P, W_{0\epsilon})\, a'(P, W_0)$$

or

$$\gamma(W_{0\epsilon} - W_0) \int_0^1 a'(P, W_0)\, c''(P, tW_{0\epsilon} + (1 - t)\, W_0)\, dt$$
$$= -\epsilon a'(P, W_0)\, a'(P, W_0).$$

By methods already developed we show that W_0 is continuous along lines parallel to the coordinate axes and we use (iv) and (v) of (IV) to show that W_0 is continuous and continuously differentiable in a neighborhood—relative to $E^2 - \{(0, 0)\}$—of the coordinate axes.

The result is established.

For each $\xi \in \mathscr{A}_h$ and for each $V \in \mathscr{H}_h^0$ let $Z = Z(\xi; V : P)$ be the solution (*Lemma* 4) of the problem

$$\begin{aligned} \gamma L_h[Z] &= \gamma L_h[\xi] - \mathscr{L}_h(V)[\xi] + f, \qquad P \in \Omega_h, \\ Z &= 0, \qquad P \in \partial\Omega_h. \end{aligned} \tag{4.11}$$

Then from (4.2) and *Lemma* 5, with $W_0 = W_0(\xi_{x\bar{x}}, \xi_{y\bar{y}} : P) = W_0(\xi)$,

$$Z(\xi; W_0 : P) \geqslant Z(\xi; V : P)$$

for all $P \in \Omega_h$ and for fall $V \in \mathscr{H}_h^0$.

In our next result we will show that $Z(\xi; W_0 : P)$ is a Lipschitz function of $\xi \in \mathscr{A}_h$.

LEMMA 13. *If the conditions in* (I), (II), *and* (IV) *are satisfied, if* $\xi_1, \xi_2 \in \mathscr{A}_h$ *with* $Z_1 = Z(\xi_1; W_0(\xi_1) : P)$ *and* $Z_2 = Z(\xi_2; W_0(\xi_2) : P)$, *then*

$$\| Z_1 - Z_2 \|_2^2 \leqslant C_2^2 \| \xi_1 - \xi_2 \|_2^2$$

with C_2^2 *given in Lemma* 6.

Proof. For each $\xi_1, \xi_2 \in \mathscr{A}_h$ let $\xi_t = t\xi_1 + (1 - t)\xi_2$, $W_{0t} = W_0(\xi_{tx\bar{x}}, \xi_{ty\bar{y}} : P)$ and $Z_t = Z(\xi_t; W_{0t} : P)$.

If $P \in \Omega_h$ is such that it is not semiplanar with respect to ξ_t for all $t \in [0, t_1)$, then the derivative of Z_t with respect to t exists and we have that

$$\gamma L_h[Z_t'] = \gamma L_h[\xi_1 - \xi_2] - \mathscr{L}_h(W_{0t})[\xi_1 - \xi_2] - \mathscr{L}_h'(W_{0t})[\xi_{0t}], \tag{4.13}$$

where $Z_t' = dZ/dt$,

$$\mathscr{L}_h'(W_{0t})[\xi_{0t}] = \{a'(P, W_{0t})\, \xi_{tx\bar{x}} + c'(P, W_{0t})\, \xi_{ty\bar{y}}\}\, dW_{0t}/dt,$$

$a' = \partial a(P, X)/\partial X$, $c' = \partial c(P, X)/\partial X$, and

$$\frac{dW_{0t}}{dt} = \frac{-a'(P, W_{0t})\, c'(P, W_{0t})}{\xi_{tx\bar{x}} K(P, W_{0t})}\, \xi_{tx\bar{x}} + \frac{c'(P, W_{0t})\, a'(P, W_{0t})}{\xi_{ty\bar{y}} K(P, W_{0t})}\, \xi_{ty\bar{y}} = 0. \tag{4.14}$$

Hence, for all such $t \in [0, t_1)$ we have that

$$\gamma L_h[Z_t'] = \gamma L_h[\xi_1 - \xi_2] - \mathscr{L}_h(W_{0t})[\xi_1 - \xi_2]. \tag{4.15}$$

At $t = t_1$ we have that $\xi_{t_1 x\bar{x}}(P) = 0$ or $\xi_{t_1 y\bar{y}}(P) = 0$. If P is not planar with respect to ξ_{t_1}—it will be semiplanar—then (4.15) remains valid. Hence we look at the case that $\xi_{t_1 x\bar{x}}(P) = \xi_{t_1 y\bar{y}}(P) = 0$. From (2.15) we have that

$$\gamma L_h[Z_{t_1} - Z_0] = \gamma L_h[\xi_{t_1} - \xi_0] - \mathscr{L}_h(W_{00})[\xi_{t_1} - \xi_0]. \tag{4.16}$$

If $P \in \Omega_h$ is such that $\xi_{2x\bar{x}}(P) = \xi_{2y\bar{y}}(P) = 0$, then $\xi_{tx\bar{x}}(P) = t\xi_{1x\bar{x}}(P)$ and $\xi_{ty\bar{y}}(P) = t\xi_{1y\bar{y}}(P)$ and we obtain (4.16) for all $t \in [0, 1]$.

If $P \in \Omega_h$ is such that $\xi_{2x\bar{x}}(P) = 0$ but $\xi_{2y\bar{y}}(P) \neq 0$, then $\xi_{tx\bar{x}}(P) = t\xi_{1x\bar{x}}(P)$ and there exists $t_1 > 0$ such that $(\xi_{tx\bar{x}}, \xi_{ty\bar{y}})$ is in a fixed quadrant for all $t \in [0, t_1)$. Now proceed as in earlier cases observing that (4.14) is valid.

As in the proof of *Lemma* 6 there exists $t_1 > 0$ such that for all $t \in [0, t_1]$

$$\| Z_t - Z_0 \| \leqslant C_2^{\,2} t_1^{\,2} \| \xi_1 - \xi_2 \|_2^2 . \tag{4.17}$$

Now continue as in that proof to obtain (4.17) for all $t \in [0, 1]$.

The next result follows the methods of Section 2.

THEOREM 7. *If the principal coefficients satisfy the conditions* (I), (II), *and* (IV) *and if* $C_2 < 1$, *then there exists a unique discrete maximal solution in* $\mathscr{S}_h$ *and a discrete maximal profile in* $\mathscr{H}_h^{\,0}$.

Let $Z_{(n)} = Z(Z_{(n-1)};\ W_0(Z_{(n-1)}) : P)$ with $Z_{(0)}$ any element of $\mathscr{S}_h$. Then, by *Lemma* 13, $Z_{(n)x\bar{x}}$ and $Z_{(n)y\bar{y}}$ will converge to $\xi_{x\bar{x}}$ and $\xi_{y\bar{y}}$ for all $P \in \Omega_h$ and ξ is the maximal solution. If $\xi_{x\bar{x}}^2(P) + \xi_{y\bar{y}}^2(P) > 0$, then $W_0(Z_{(n)} : P) \to W_0(\xi : P)$ as $n \to \infty$ by *Lemma* 12. If $\xi_{x\bar{x}}^2(P) + \xi_{y\bar{y}}^2(P) = 0$, then

$$a(P,\ W_0(Z_{(n-1)} : P))\ Z_{(n-1)x\bar{x}}(P) \to a(P,\ W_0(\xi : P))\ \xi_{x\bar{x}}(P)$$

and

$$a(P,\ W_0(Z_{(n-1)} : P))\ Z_{(n-1)y\bar{y}}(P) \to a(P,\ W_0(\xi : P))\ \xi_{y\bar{y}}(P) \quad \text{as} \quad n \to \infty.$$

Hence we would have the convergence of $W_0(Z_{(n-1)} : P)$ to $W_0(\xi : P)$ at all points where $|f(P)| > 0$.

An analysis of the differential equation in (1.1) when the conditions (I), (II), and (IV) are satisfied by the principal coefficients proceeds along lines similar to that taken in Section 3 with obvious modifications already presented in this section.

5. SOME NUMERICAL EXAMPLES

Let $\Omega = (0, 1) \times (0, 1)$, $[A, B] = [\pi/2, 3\pi/2]$, $a(P, X) = 1 + \epsilon \sin X$, $b(P, X) = 0$, and $c(P, X) \equiv 1$ with $0 < \epsilon < 1$. The conditions (I), (II), and (III) are satisfied.

Let $\epsilon = 0.1$, $\alpha = 0$, $f(P) \equiv -1$, and $\gamma = 1$. Then the hypotheses of *Theorem* 1 are satisfied. Choose the mesh size $h = 0.25$. Computations were performed in double precision on an IBM370-168. The iteration process was terminated when

$$\| Z_{(n+1)} - Z_{(n)} \| / \| Z_{(n+1)} \| < 10^{-8}. \tag{5.1}$$

The scheme converged in eight iterations with the maximal profile given as the constant function $W_0(P) = 3\pi/2$. The corresponding maximal solution is given in *Table* I with $U_{ij} = U(ih, jh)$. Due to symmetry we need only list the values as given. In *Table* I we also give U'_{ij}, where U'_{ij} is the solution corresponding to the profile $3\pi/2 + \delta(P'; P)$ with $P' = (1/2, 1/2)$. Although we do not list it in *Table* I we have compared U_{ij} with all possible solutions and found that U_{ij} is indeed maximal.

TABLE I

i	j	U_{ij}	U'_{ij}
1	1	0.04523597	0.04514034
1	2	0.05778250	0.05759604
2	1	0.05734908	0.05715252
2	2	0.07400175	0.07342695

We have applied the methods of *Theorem* 7 to the Eq. (2.1) with $a(P, X) = 2 + \sin X$, $b(P, X) \equiv 0$, $c(P, X) = 2 + \cos X$, $\Omega = (0, 1) \times (0, 1)$, $[A, B] = [0, 2\pi]$ and $h = 0.25$. When $\alpha = 0$ and $\gamma = 1$ the convergence criterion in (5.1) was satisfied in nineteen iterations.

In Table II we list the values U_{ij} of the discrete maximal solution and the values W_{0ij} of a discrete maximal profile.

TABLE II

i	j	U_{ij}	W_{0ij}
1	1	0.03343477	3.92698956
1	2	0.04269895	4.16812992
2	1	0.04269895	3.68584919
2	2	0.05478425	3.92698956

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